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The matrix realization of affine Jacobi varieties and the extended Lotka–Volterra lattice

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Abstract

We study completely integrable Hamiltonian systems whose monodromy matrices are related to the representatives for the set of gauge equivalence classes $\mathcal{M}_F$ of polynomial matrices. Let X be the algebraic curve given by the common characteristic equation for $\mathcal{M}_F$. We construct the isomorphism from the set of representatives to an affine part of the Jacobi variety of X. This variety corresponds to the invariant manifold of the system, where the Hamiltonian flow is linearized. As an application, we discuss the algebraic complete integrability of the extended Lotka–Volterra lattice with a periodic boundary condition.

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1. Introduction

The algebro-geometric structure of the completely integrable Hamiltonian systems was unveiled around 1980 (see [1–5] and references therein), and has been extensively studied. It was a remarkable discovery that the Hamiltonian flows of the systems are linearized on algebraic varieties such as the Jacobi variety $J(X)$ of an algebraic curve X. Many of the systems are described by the Lax equation for (Laurent) polynomial matrices of a spectral parameter, and X comes from its fixed characteristic equation which gives the level set of the Lax matrix. Typically the flows are linearized by the following procedure:

$$\begin{aligned} \text{System} &\xrightarrow{\text{(I)}} \text{Lax matrix } (\to X) \\ &\xrightarrow{\text{(II)}} \mathrm{Div}_{\mathrm{eff}}(X) \\ &\xrightarrow{\text{(III)}} J(X) \end{aligned} \tag{1.1}$$

where $\mathrm{Div}_{\mathrm{eff}}(X)$ is the set of effective divisors. The arrows (II) and (III) are, respectively, induced by the eigenvector map and the Abel map. In many cases, the linearization of the

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flows are related to the Lie algebraic symmetry of Lax matrices [5]. On the other hand, in [6] the condition of the linearization was discussed based on a cohomological interpretation of the Lax equation.

Roughly speaking, (I) is heuristic, and (II) and (III) are systematic. By the Abel–Jacobi theorem (III) is understood in general framework, but (II) depends on the Lax matrix very much. Beauville showed that if we replace the Lax matrix with a set of gauge equivalence classes of polynomial matrices, (II) becomes an isomorphism [7]. He further proved that over the tangent space of the set there exists the g-dimensional invariant vector field linearized on $J(X)$, where g is the genus of X. As claimed in [8], to study concrete integrable systems we need to choose the orbit which gives the representative of the gauge equivalence class. In fact, Mumford already gave an important example when X was a hyperelliptic curve [9], and introduced the set of representatives with the explicit isomorphic maps (II) and (III). The dynamical system that he introduced is called the Mumford system, and has been studied from many points of view [10–14]. Recently, Smirnov and Zeitlin constructed the representative of the wider class of gauge equivalence classes, by starting with N by N monodromy matrices of some special forms [8, 15]. They constructed the isomorphism (II) by making use of the separation of variables (SoV) in the style of Sklyanin [16].

In this paper, we consider the extension of [15], and construct the isomorphic map (II) for a certain class of monodromy matrices. We introduce N by N monodromy matrices $\mathbf{T}_{m;n_1,n_2}(z)(n_1 = 1, \ldots, N-1, n_2 = 1, \ldots, N)$, whose entries are polynomials of a spectral parameter z of degree m. We fix a level set of $\mathbf{T}_{m;n_1,n_2}(z)$, where the characteristic polynomial of $\mathbf{T}_{m;n_1,n_2}(z)$ is fixed to be $F_{m;n_1,n_2}(z, w) \in \mathbb{C}[z, w]$. We write this set as $\{\mathbf{T}_{m;n_1,n_2}(z)\}_{F_{m;n_1,n_2}}$. The characteristic equation $F_{m;n_1,n_2}(z, w) = 0$ defines a complete algebraic curve X and the set of gauge equivalence classes $\mathcal{M}_{F_{m;n_1,n_2}}$. Let $\{\mathbf{M}(z)\}_{F_{m;n_1,n_2}}$ be the set of representatives of $\mathcal{M}_{F_{m;n_1,n_2}}$. Starting with the level set $\left\{\mathbf{T}_{m;n_1,n_2}(z)\right\}_{F_{m;n_1,n_2}}$, we study the following diagram

$$\begin{array}{ccc} \left\{\mathbf{T}_{m;n_1,n_2}(z)\right\}_{F_{m;n_1,n_2}} & \xrightarrow{\text{(b)}} & X(g) \\ \text{(a)} \downarrow & & \nearrow \text{(c)} \\ \{\mathbf{M}(z)\}_{F_{m;n_1,n_2}} & & \end{array} \tag{1.2}$$

where $X(g) \subset \mathrm{Div}_{\mathrm{eff}}(X)$ is the set of effective divisors of degree g. The map (a) is the gauge transformation, (b) is based on SoV and we construct these two so as to make the diagram (1.2) commutative. In [15], the maps in (1.2) were given for $\{\mathbf{T}_{m;1,1}(z)\}_{F_{m;1,1}}$ for a general N. We study (1.2) in detail for $N = 2$ and 3 here.

Next, as an application, we study the integrable Hamiltonian structure of the extended Lotka–Volterra lattice. This is defined by the differential-difference equation

$$\frac{\mathrm{d}V_n}{\mathrm{d}t} = 2V_n \sum_{k=1}^{N-1} (V_{n+k} - V_{n-k}) \tag{1.3}$$

where $V_n \equiv V_n(t) \in \mathbb{C}$ for $n \in \mathbb{Z}$. This model has the Hamiltonian structure and a family of integrals of motion in involution [17–19]. When the model is infinite dimensional, the $N = 2$ case is known as the lattice KdV hierarchy [20], and the general $N > 2$ case is related to the lattice N-reduced KP hierarchy [21–24]. We set a periodic boundary condition $V_{n+L} = V_n$ for $L \in \mathbb{Z}_{\geqslant 2N-1}$, and write $\mathrm{LV}(N, L)$ for this finite-dimensional model. In [25] the algebraic complete integrability of $\mathrm{LV}(2, L)$ was shown, based on the analogues of the Mumford system, and its invariant manifold is associated with an affine part of the Prym variety. Now, following [26], we study the integrability of $\mathrm{LV}(N, L)$ by applying the structure (1.2). We show that the monodromy matrix of $\mathrm{LV}(N, L)$ is related to $\mathbf{T}_{m;n_1,n_2}(z)$ where the correspondence $L \leftrightarrow (m, n_1, n_2)$ is determined uniquely, and that the Poisson structure over

$\{\mathbf{M}(z)\}_{F_{m;n_1,n_2}}$ is nicely embedded in that of $\mathrm{LV}(N, L)$. Thus SoV can be used to describe explicitly the map (c) as algebraic relations between the divisors in $X(g)$ and the dynamical variables V_n. Finally we give another proof of the algebraic complete integrability for $N = 2$ case, and establish it for $N = 3$ case.

Theorem 1.1. *$LV(N, L)$ is algebraic completely integrable for $L \in \mathbb{Z}_{\geqslant 2N-1}$, $N = 2$ and 3.*

We believe that it is true for general N.

This paper is arranged as follows: in section 2, after preliminaries we introduce a class of N by N monodromy matrices $\mathbf{T}_{m;n_1,n_2}(z)$ which satisfy the fundamental Poisson relation with the classical r-matrix. By starting with these matrices we explain how to construct the maps in (1.2). In section 3, we study the $N = 2, 3$ cases, where the set of representatives $\{\mathbf{M}(z)\}_{F_{m;n_1,n_2}}$ and the eigenvector map (c) (1.2) are explicitly obtained. In section 4, we discuss the Hamiltonian structure of $\mathrm{LV}(N, L)$ and prove theorem 1.1.

The advantage of our way to investigate $\mathrm{LV}(N, L)$ is that we obtain the isomorphic eigenvector map explicitly written as algebraic relations between the divisor and the dynamical variables. On the other hand, as discussed in [25], for a model given by homogeneous evolution equations such as (1.3), the Painlevé analysis [27] becomes a powerful tool to construct the associated invariant manifold. It may be interesting to study the invariant manifold for $\mathrm{LV}(N, L)$ from these two viewpoints.

2. Representatives for $\mathcal{M}_F$ and eigenvector map

2.1. Preliminaries

Fix a polynomial $F(z, w)$ of the form

$$F(z, w) \equiv w^N - f_1(z)w^{N-1} + f_2(z)w^{N-2} - \cdots + (-1)^N f_N(z) \tag{2.1}$$

where each polynomial $f_i(z)$ satisfies $\deg f_i(z) \leqslant im$. Let X be the complete algebraic curve defined by $F(z, w) = 0$. We assume X is smooth, and let g be its genus. Let $\mathcal{M}_F$ be the set of gauge equivalence classes of N by N matrices whose matrix elements are polynomials in z of degree $m \in \mathbb{Z}_{>0}$,

$$\begin{aligned}\mathcal{M}_F = \{\mathbf{M}(z) \mid \deg(\mathbf{M}(z)_{i,j}) \leqslant m \qquad &\text{for all } i, j \\ \mathrm{Det}(w\mathbb{1} - \mathbf{M}(z)) &= F(z, w)\}/\mathbf{GL}_N(\mathbb{C}).\end{aligned} \tag{2.2}$$

For $\mathcal{M}_F$ Beauville introduced the isomorphism [7]

$$\mathcal{M}_F \simeq X(g) - D. \tag{2.3}$$

Here $X(g)$ is the set of effective divisors $X(g) = X^g/\mathfrak{S}_g \subset \mathrm{Div}_{\mathrm{eff}}(X)$, $\mathfrak{S}_g$ is the symmetric group and D is a subset of $X(g)$. The Abel map induces the isomorphism,

$$X(g) - D \simeq J(X) - \Theta \tag{2.4}$$

where D is mapped to a $(g-1)$-dimensional subvariety Θ called the theta divisor of the Jacobi variety $J(X)$.

We call $J(X) - \Theta$ the affine Jacobi variety of X and write $J_{\mathrm{aff}}(X)$ for it. We denote the set of representatives of $\mathcal{M}_F$ using $\{\mathbf{M}(z)\}_F$. Due to (2.3) and (2.4) $\{\mathbf{M}(z)\}_F$ gives the matrix realization of $J_{\mathrm{aff}}(X)$. Herewith the arrows (II) and (III) in the procedure (1.1) become isomorphisms (II′) and (III′),

$$\begin{aligned}\{\mathbf{M}(z)\}_F &\xrightarrow{(\mathrm{II}')} X(g) - D \\ &\xrightarrow{(\mathrm{III}')} J_{\mathrm{aff}}(X).\end{aligned} \tag{2.5}$$

In this paper, we let $\mathbf{M}_N(\mathbb{C})$ be a set of N by N complex matrices, $\mathbf{E}_{i,j}$ be an N by N basic matrix: $(\mathbf{E}_{i,j})_{m,n} = \delta_{m,i}\delta_{n,j}$, and $\vec{e}_i$ be an N-dimensional row vector: $(\vec{e}_i)_m = \delta_{m,i}$.

2.2. *Classification of monodromy matrices and $\mathcal{M}_F$*

We introduce lower/upper triangular N by N matrices,

$$
\boldsymbol{\mu}_-^{(i)} = \begin{array}{r} \\ \\ \\ i+1\rightarrow \\ \\ \\ \\ \\ \end{array}
\begin{array}{c}
\begin{pmatrix}
0 & \cdots & & & & & \cdots & 0 \\
\vdots & & & & & & & \vdots \\
0 & \cdots & & & & & \cdots & 0 \\
* & * & 0 & \cdots & & & \cdots & 0 \\
* & * & * & 0 & \cdots & & \cdots & 0 \\
\vdots & & & \ddots & \ddots & & & \vdots \\
* & \cdots & \cdots & \cdots & * & 0 & \cdots & 0
\end{pmatrix} \\
\qquad\qquad\qquad \uparrow N+1-i
\end{array}
\qquad \text{for} \quad i = 1, \ldots, N-1
$$

$$
\boldsymbol{\mu}_+^{(i)} =
\begin{array}{c}
\qquad\quad \downarrow i \\
\begin{pmatrix}
0 & \cdots & 0 & * & & \cdots & \cdots & * \\
\vdots & & & \ddots & * & & \cdots & * \\
\vdots & & & & \ddots & & \ddots & \vdots \\
0 & \cdots & & & & \cdots & 0 & * \\
0 & \cdots & & & & & \cdots & 0 \\
\vdots & & & & & & & \vdots \\
0 & \cdots & & & & & \cdots & 0
\end{pmatrix}
\end{array}
\begin{array}{l} \\ \\ \\ \\ \leftarrow N+1-i \\ \\ \\ \\ \end{array}
\qquad \text{for} \quad i = 1, \ldots, N
\tag{2.6}
$$

where $i\rightarrow$ (or $\downarrow i$) indicates the ith row (or column) of the matrices, and $*$ denotes non-zero entries which will be constants or variables. For $N \geqslant 3$ we also use

$$
\boldsymbol{\mu}_-^{(-i)} =
\begin{array}{c}
\qquad \downarrow i+2 \\
\begin{pmatrix}
* & \cdots & * & 0 & \cdots & 0 \\
* & \cdots & \cdots & * & \ddots & \vdots \\
\vdots & & & & \ddots & 0 \\
* & \cdots & & & \cdots & * \\
* & \cdots & & & \cdots & * \\
\vdots & & & & & \vdots \\
* & \cdots & \cdots & \cdots & \cdots & *
\end{pmatrix}
\end{array}
\begin{array}{l} \\ \\ \\ \\ \\ \leftarrow N-1-i \\ \\ \\ \\ \end{array}
\qquad \text{for} \quad i = 0, \ldots, N-3
$$

$$
\boldsymbol{\mu}_+^{(-i)} = \begin{array}{r} \\ \\ \\ \\ i+2\rightarrow \\ \\ \\ \\ \\ \end{array}
\begin{array}{c}
\begin{pmatrix}
* & \cdots & \cdots & \cdots & \cdots & * \\
\vdots & & & & & \vdots \\
* & \cdots & & & \cdots & * \\
* & \cdots & & & \cdots & * \\
0 & \ddots & & & & \vdots \\
\vdots & \ddots & * & \cdots & \cdots & * \\
0 & \cdots & 0 & * & \cdots & *
\end{pmatrix} \\
\qquad\quad \uparrow N-i-1
\end{array}
\qquad \text{for} \quad i = 0, \ldots, N-3.
\tag{2.7}
$$

We denote by $\boldsymbol{\mu}_j$ $(j \in \mathbb{Z}_{>0})$ N by N matrices whose entries are not identically zero. We write $\big(\boldsymbol{\mu}_-^{(i)} \cap \boldsymbol{\mu}_+^{(j)}\big)$ for a matrix which has zero at (j_1, j_2) if $\big(\boldsymbol{\mu}_-^{(i)}\big)_{j_1,j_2}$ or $\big(\boldsymbol{\mu}_+^{(j)}\big)_{j_1,j_2}$ is zero. Note that $\big(\boldsymbol{\mu}_-^{(i)} \cap \boldsymbol{\mu}_j\big)$ and $\boldsymbol{\mu}_-^{(i)}$ have the same form.

First we fix the matrices (2.6), (2.7) and $\boldsymbol{\mu}_j$ for $j = 1, \ldots, m-1$ to be constant matrices in $\mathbf{M}_N(\mathbb{C})$: $\boldsymbol{\mu}_-^{(i)} \equiv \boldsymbol{\mu}_-^{(i)0}$, $\boldsymbol{\mu}_+^{(i)} \equiv \boldsymbol{\mu}_+^{(i)0}$ and $\boldsymbol{\mu}_j \equiv \boldsymbol{\mu}_j^0$. Using these matrices we define polynomial matrices $\mathbf{T}^0_{m;n_1,n_2}(z)(m \in \mathbb{Z}_{>0}, n_1 \in \{1, 2, \ldots, N-1\}, n_2 \in \{1, 2, \cdots, N\})$ as

$$\mathbf{T}^0_{m;n_1,n_2}(z) = \begin{cases} \boldsymbol{\mu}_-^{(n_1)0} z^m + \boldsymbol{\mu}_-^{(n_1-N+1)0} z^{m-1} + \boldsymbol{\mu}_2^0 z^{m-2} + \cdots + \boldsymbol{\mu}_{m-2}^0 z^2 & \\ \quad + \boldsymbol{\mu}_+^{(n_2-N)0} z + \boldsymbol{\mu}_+^{(n_2)0} & \text{for} \quad m \geqslant 3 \\ \boldsymbol{\mu}_-^{(n_1)0} z^2 + \big(\boldsymbol{\mu}_-^{(n_1-N+1)0} \cap \boldsymbol{\mu}_+^{(n_2-N)0}\big) z + \boldsymbol{\mu}_+^{(n_2)0} & \text{for} \quad m = 2 \\ \big(\boldsymbol{\mu}_-^{(n_1)0} \cap \boldsymbol{\mu}_+^{(n_2-N)0}\big) z + \big(\boldsymbol{\mu}_-^{(n_1-N+1)0} \cap \boldsymbol{\mu}_+^{(n_2)0}\big) & \text{for} \quad m = 1. \end{cases} \tag{2.8}$$

When $\boldsymbol{\mu}_-^{(n_1-N+1)}$ (or $\boldsymbol{\mu}_+^{(n_2-N)}$) is not defined by (2.7), set $\boldsymbol{\mu}_-^{(n_1-N+1)} \equiv \boldsymbol{\mu}_1^0$ (or $\boldsymbol{\mu}_+^{(n_2-N)} \equiv \boldsymbol{\mu}_{m-1}^0$).

Proposition 2.1. *The map*

$$\begin{aligned} &\mathbb{Z}_{>0} \times \{1, 2, \ldots, N-1\} \times \{1, 2, \ldots, N\} \to \mathbb{C}[z, w] \\ &(m, n_1, n_2) \mapsto F_{m;n_1,n_2}(z, w) = \mathrm{Det}\big(w\mathbb{1} - \mathbf{T}^0_{m;n_1,n_2}(z)\big) \end{aligned} \tag{2.9}$$

is injective.

Proof. It is sufficient to check the coefficient of w in the polynomial $F_{m;n_1,n_2}(z, w)$ (2.9) for the injectivity. We write $f_{N-1}(z)$ for this coefficient, as $F_{m;n_1,n_2}(z, w)$ (2.9) has a form (2.1). Note

$$f_{N-1}(z) = \mathrm{Det}\, \mathbf{T}^0_{m;n_1,n_2}(z) \cdot \mathrm{Tr}\big(\mathbf{T}^0_{m;n_1,n_2}(\mathrm{z})^{-1}\big)$$

and the forms of $\boldsymbol{\mu}_+^{(n_1)}$ and $\boldsymbol{\mu}_-^{(n_2)}$ which compose $\mathbf{T}^0_{m;n_1,n_2}(z)$. Then one sees

$$\deg f_{N-1}(z) = (N-1)m - n_1 + 1 \qquad \mathrm{ord}_{z=0} f_{N-1}(z) = n_2 - 1.$$

Since $n_1 \in \{1, \ldots, N-1\}$, $f_{N-1}(z)$ determines a triple (m, n_1, n_2). □

Next we set the entries of matrices (2.6), (2.7) and $\boldsymbol{\mu}_j$ for $j = 1, \ldots, m-1$ to be variables, and define N by N monodromy matrices $\mathbf{T}_{m;n_1,n_2}(z)$ $(m \in \mathbb{Z}_{>0}, n_1 \in \{1, 2, \ldots, N-1\}, n_2 \in \{1, 2, \ldots, N\})$ as in (2.8),

$$\mathbf{T}_{m;n_1,n_2}(z) = \begin{cases} \boldsymbol{\mu}_-^{(n_1)} z^m + \boldsymbol{\mu}_-^{(n_1-N+1)} z^{m-1} + \boldsymbol{\mu}_2 z^{m-2} + \cdots + \boldsymbol{\mu}_{m-2} z^2 & \\ \quad + \boldsymbol{\mu}_+^{(n_2-N)} z + \boldsymbol{\mu}_+^{(n_2)} & \text{for} \quad m \geqslant 3 \\ \boldsymbol{\mu}_-^{(n_1)} z^2 + \big(\boldsymbol{\mu}_-^{(n_1-N+1)} \cap \boldsymbol{\mu}_+^{(n_2-N)}\big) z + \boldsymbol{\mu}_+^{(n_2)} & \text{for} \quad m = 2 \\ \big(\boldsymbol{\mu}_-^{(n_1)} \cap \boldsymbol{\mu}_+^{(n_2-N)}\big) z + \big(\boldsymbol{\mu}_-^{(n_1-N+1)} \cap \boldsymbol{\mu}_+^{(n_2)}\big) & \text{for} \quad m = 1. \end{cases} \tag{2.10}$$

To study $\mathbf{T}_{m;n_1,n_2}(z)$, we define a local Lax matrix as

$$\mathbf{L}_n(z) = \sum_{k=1}^{N-1} l_n^{(k)} \mathbf{E}_{k,k+1} + z l_n^{(N)} \mathbf{E}_{N,1} + z l_n^{(0)} \mathbf{E}_{N,2} \tag{2.11}$$

where $l_n^{(k)} (n \in \mathbb{Z}, k = 0, \ldots, N)$ are dynamical variables.

Lemma 2.2. *With the Lax matrix $\mathbf{L}_n(z)$ (2.11) the following Poisson relation is compatible,*

$$\{\mathbf{L}_n(z) \overset{\otimes}{,} \mathbf{L}_m(z')\} = \delta_{n,m} [\mathbf{r}(z/z'), \mathbf{L}_n(z) \otimes \mathbf{L}_n(z')] \tag{2.12}$$

where $\mathbf{r}(z)$ *is the classical r-matrix*

$$\mathbf{r}(z) = \frac{z+1}{z-1}\sum_{k=1}^{N}\mathbf{E}_{k,k}\otimes\mathbf{E}_{k,k} + \frac{2}{z-1}\sum_{1\leqslant j<k\leqslant N}(z\mathbf{E}_{k,j}\otimes\mathbf{E}_{j,k} + \mathbf{E}_{j,k}\otimes\mathbf{E}_{k,j}). \tag{2.13}$$

Proof. It is shown by a direct calculation. One easily sees that (2.12) is consistent with the Poisson bracket algebra for $l_n^{(k)} (k = 0, \ldots, N)$ defined as

$$\left\{l_n^{(k)}, l_m^{(j)}\right\} = 0 \qquad \text{for} \quad 1 \leqslant k, j \leqslant N$$
$$\left\{l_n^{(0)}, l_m^{(k)}\right\} = \delta_{n,m}(\delta_{k,N} - \delta_{k,1})l_n^{(0)}l_n^{(k)}.$$

□

We define integers m, m_1 and m_2 by

$$m = \left[\frac{L}{N(N-1)}\right] \qquad m_1 = \left[\frac{L}{N-1}\right] \qquad m_2 = \left[\frac{L}{N}\right] \tag{2.14}$$

and determine k, k_1 and k_2 using

$$L = (N-1)m_1 + k_1 = Nm_2 + k_2 = N(N-1)m + k. \tag{2.15}$$

Lemma 2.3. *The monodromy matrix* $\mathbf{T}_{m;n_1,n_2}(z)$ *(2.10) can be written as a product of L Lax matrices* $\mathbf{L}_n(z)$ *(2.11),*

$$z^{-m_2}\prod_{n=1}^{L}\mathbf{L}_n(z) = \begin{cases}\mathbf{T}_{m;1,1}(z) & \textit{for} \quad k_1 = k_2 = 0\\ \mathbf{T}_{m+1;N-k_1,k_2+1}(z) & \textit{for} \quad k_1, k_2 \neq 0 \quad 0 \leqslant k_1 - k_2 \leqslant N-2\\ \mathbf{T}_{m+2;N-k_1,k_2+1}(z) & \textit{for} \quad k_1 - k_2 \leqslant -1\end{cases} \tag{2.16}$$

where integers m, m_2, k_1 *and* k_2 *are defined in (2.14) and (2.15).*

See appendix A for the proof. Due to lemmas 2.2 and 2.3, it is straightforward to obtain the following proposition.

Proposition 2.4. *With the matrix* $\mathbf{T}_{m;n_1,n_2}(z)$ *the fundamental Poisson relation is compatible,*

$$\left\{\mathbf{T}_{m;n_1,n_2}(z) \overset{\otimes}{,} \mathbf{T}_{m;n_1,n_2}(z')\right\} = \left[\mathbf{r}(z/z'), \mathbf{T}_{m;n_1,n_2}(z)\otimes\mathbf{T}_{m;n_1,n_2}(z')\right]. \tag{2.17}$$

Let $\mathcal{A}_C$ be the Poisson bracket algebra over the polynomial ring generated by the coefficients of entries in $\mathbf{T}_{m;n_1,n_2}(z)$, whose defining relation is (2.17). Then (2.17) implies

Proposition 2.5 [28]**.***(i) The determinant of* $\mathbf{T}_{m;n_1,n_2}(z)$ *belongs to the centre of* $\mathcal{A}_C$*,*

$$\left\{\mathbf{T}_{m;n_1,n_2}(z), \operatorname{Det}\mathbf{T}_{m;n_1,n_2}(z')\right\} = 0.$$

(ii) The coefficients of the characteristic polynomial of $\mathbf{T}_{m;n_1,n_2}(z)$ *are Poisson commutative,*

$$\{\operatorname{Det}(w\mathbb{1} - \mathbf{T}_{m;n_1,n_2}(z)), \operatorname{Det}(w'\mathbb{1} - \mathbf{T}_{m;n_1,n_2}(z'))\} = 0.$$

Using (2.9) we define the level set of $\mathbf{T}_{m;n_1,n_2}(z)$ as

$$\left\{\mathbf{T}_{m;n_1,n_2}(z)\right\}_{F_{m;n_1,n_2}} = \left\{\mathbf{T}_{m;n_1,n_2}(z)\middle|\operatorname{Det}(w\mathbb{1} - \mathbf{T}_{m;n_1,n_2}(z)) = F_{m;n_1,n_2}(z,w)\right\}.$$

Let X be the complete algebraic curve determined by $F_{m;n_1,n_2}(z,w) = 0$, and its genus be g. We consider the cases of $g \geqslant 1$. In general $\left\{\mathbf{T}_{m;n_1,n_2}(z)\right\}_{F_{m;n_1,n_2}}$ constitutes a variety whose dimension is greater than g. Since the isomorphism (2.3) implies that $\mathcal{M}_{F_{m;n_1,n_2}}$ is a g-dimensional variety, we state a problem to construct the map (a) (1.2) which gives the set of representatives $\{\mathbf{M}(z)\}_{F_{m;n_1,n_2}}$ as follows:

Problem 2.6. *For* $\mathbf{T}_{m;n_1,n_2}(z)$ *find a gauge matrix* $\mathbf{S}$ *on* $\mathbf{T}_{m;n_1,n_2}(z)$*, such that the set*

$$\{\mathbf{M}(z)\}_{F_{m;n_1,n_2}} = \{\mathbf{M}(z) = \mathbf{S}\mathbf{T}_{m;n_1,n_2}(z)\mathbf{S}^{-1} \mid \mathrm{Det}(w\mathbb{1} - \mathbf{M}(z)) = F_{m;n_1,n_2}(z, w)\} \tag{2.18}$$

constitutes a g-dimensional variety.

We note that the matrix $\mathbf{M}(z)$ has the same degree as $\mathbf{T}_{m;n_1,n_2}(z)$ as a polynomial matrix, and write it as

$$\mathbf{M}(z) = \boldsymbol{\eta}_0 z^m + \boldsymbol{\eta}_1 z^{m-1} + \cdots + \boldsymbol{\eta}_{m-1} z + \boldsymbol{\eta}_m. \tag{2.19}$$

Here the variable matrices $\boldsymbol{\eta}_i$ do not depend on z. Once the above problem is solved, the Poisson bracket algebra generated by the matrix elements of $\boldsymbol{\eta}_i$ (2.19) is induced by $\mathcal{A}_C$, and we let $\mathcal{A}_M$ be this algebra. Due to proposition 2.5, the coefficients of $\mathrm{Det}\big(w\mathbb{1} - \mathbf{T}_{m;n_1,n_2}(z)\big)$ constitute the commuting subalgebra of $\mathcal{A}_M$.

In what follows, we abbreviate the polynomial $F_{m;n_1,n_2}(z, w)$ to $F(z, w)$, and the subscript $_{F_{m;n_1,n_2}}$ to $_F$.

2.3. Eigenvector map and SoV

Following [8, 15] we introduce the eigenvector map (b) (1.2) by making use of SoV. Sklyanin refined the technique invented to solve the spectral problem of the quantum Toda lattice, and introduced the method called SoV based on the R-matrix structure of the monodromy matrices (see [16, 29] and references therein). The SoV for the monodromy matrices of $SL(N)$ symmetry has been studied in detail. The cases of $N = 2$ and 3 are done by Sklyanin himself [29, 30], and the extension to the general N cases are clarified in [31, 32].

For classical systems this method derives the canonically conjugate variables from the poles of the eigenvector of the monodromy matrix. We review this mechanism following [16]. Let $\mathbf{T}(z)$ be an N by N monodromy matrix which satisfies the fundamental Poisson relation as (2.17). Then the eigenvector of $\mathbf{T}(z)$ called the Baker–Akhiezer function is defined as

$$\mathbf{T}(z)\vec{\phi}(z) = w\vec{\phi}(z) \qquad \sum_{n=1}^{N} a_n(z)\phi_n(z) = 1$$

where $\vec{\phi}(z) = (\phi_1(z), \ldots, \phi_N(z))$, and w is the eigenvalue. The second equation is a normalization which uniquely determines $\vec{\phi}(z)$. When $\vec{\phi}(z)$ has a pole at $z = z_i$, the residues $\vec{\phi}_i = (\phi_{1,i}, \ldots, \phi_{N,i}) = \mathrm{res}_{z=z_i}\vec{\phi}(z)$ satisfy

$$\mathbf{T}(z_i)\vec{\phi}_i = w_i\vec{\phi}_i \qquad \sum_{n=1}^{N} a_n(z_i)\phi_{n,i} = 0. \tag{2.20}$$

Then the condition to get non-zero vector $\vec{\phi}_i$ becomes

$$\mathrm{Det}\begin{pmatrix} a_1(z) & a_2(z) & \cdots & a_N(z) \\ T(z)_{1,1} - w & T(z)_{1,2} & \cdots & T(z)_{1,N} \\ \vdots & & & \\ T(z)_{j-1,1} & T(z)_{j-1,2} & \cdots & T(z)_{j-1,N} \\ T(z)_{j+1,1} & T(z)_{j+1,2} & \cdots & T(z)_{j+1,N} \\ \vdots & & & \\ T(z)_{N,1} & T(z)_{N,2} & \cdots & T(z)_{N,N} - w \end{pmatrix} = 0 \qquad \text{for} \quad j = 1, \ldots N \tag{2.21}$$

where $T(z)_{i,j} = (\mathbf{T}(z))_{i,j}$.

In our case with the monodromy matrix $\mathbf{T}_{m;n_1,n_2}(z)$ (2.10), some simple choices of the vector $\vec{a}(z) = (a_1(z), \ldots, a_N(z))$ give SoV, and (2.21) reduces to two equations on $\mathbf{T}_{m;n_1,n_2}(z)$,

$$B(z) = 0 \qquad w = A(z). \tag{2.22}$$

Here $A(z) = A\big(\mathbf{T}_{m;n_1,n_2}(z)\big)$ is a rational function of z, and $B(z) = B\big(\mathbf{T}_{m;n_1,n_2}(z)\big)$ is a polynomial. Accordingly a zero z_i of $B(z)$ uniquely determines the eigenvalue $w_i = A(z_i)$. The significant benefit of the fundamental Poisson relation (2.17) is that the variables (z_i, w_i) turn out to be canonically conjugate variables, namely they fulfil the canonical Poisson brackets,

$$\{z_i, z_j\} = \{w_i, w_j\} = 0 \qquad \{z_i, w_j\} = 2\delta_{i,j} z_i w_i.$$

These variables are called the *separated variables*, and the equation $B(z) = 0$ (2.22) is called the separation equation.

When we consider the level set $\big\{\mathbf{T}_{m;n_1,n_2}(z)\big\}_F$, each pair (z_i, w_i) satisfies $F(z_i, w_i) = 0$. We expect that the separation equation has a following form,

$$B(z) = B_0 z^{f(n_1,n_2)} \prod_{i=1}^{g} (z - z_i) \qquad f(n_1, n_2) \in \mathbb{Z}_{\geqslant 0} \tag{2.23}$$

where g is the genus of the algebraic curve X given by $F(z, w) = 0$. There are certainly some different choices of the separation equations (2.22) depending on the vector $\vec{a}(z)$. To make the diagram (1.2) commutative, we should choose the separation equation invariant under the gauge $\mathbf{S}$ (2.18).

To close this section, we mention the subset D which appeared in the isomorphism (2.4). We assume $(z_i, w_i) \neq (z_j, w_j)$ for $i \neq j$, and a set of the g-separated variables (z_i, w_i) determine an effective divisor

$$P = \sum_{i=1}^{g} [(z_i, w_i)] \in X(g). \tag{2.24}$$

Then the subset D should be set as [15]

$$D = \left\{ P = \sum_{i=1}^{g} [(z_i, w_i)] \mid \mathrm{Det}(h_i(z_j, w_j))_{1\leqslant i,j\leqslant g} = 0 \right\} \tag{2.25}$$

where $h_i(z, w)$ are defined by homomorphic 1-forms σ_i on X [33],

$$\sigma_i(z, w) = \frac{h_i(z, w)\,\mathrm{d}z}{\frac{\partial}{\partial w} F(z, w)} \qquad \text{for} \quad i = 1, \ldots, g. \tag{2.26}$$

We remark that the g independent vector fields on a tangent space of $\mathcal{M}_F$ are generated by the coefficients of $F(z, w)$ (2.9). The fundamental Poisson relation (2.17) ensures that the evolution of the divisor P generated by the vector fields is linearized on $J_{\mathrm{aff}}(X)$.

3. Study of concrete cases

Starting with $\mathbf{T}_{m;n_1,n_2}(z)$, we study the diagram (1.2). We construct the gauge matrix $\mathbf{S}$ (2.18) which gives the set of representatives $\{\mathbf{M}(z)\}_F$, and the associated separation equation which makes the map (c) (1.2) well defined. Then the isomorphic eigenvector map (II′) (2.5) is induced by (c). We explicitly discuss the cases of $N = 2$ and 3 with $g \geqslant 1$. Further we recall $\{\mathbf{M}(z)\}_F$ associated with $\mathbf{T}_{m;1,1}(z)$ for general N [15, 26].

3.1. $N=2$ case

We have matrices (2.6)

$$\boldsymbol{\mu}_-^{(1)} = \begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix} \qquad \boldsymbol{\mu}_+^{(1)} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \qquad \boldsymbol{\mu}_+^{(2)} = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}.$$

Using them we introduce two matrices, $\mathbf{T}_{m;1,1}(z)$ and $\mathbf{T}_{m;1,2}(z)$, and derive the associated representatives:

(i) $\mathbf{T}_{m;1,1}(z)$. We have the matrix

$$\mathbf{T}_{m;1,1}(z) = \boldsymbol{\mu}_-^{(1)} z^m + \boldsymbol{\mu}_1 z^{m-1} + \cdots + \boldsymbol{\mu}_{m-1} z + \boldsymbol{\mu}_+^{(1)} \qquad \text{for} \quad m \geqslant 2. \tag{3.1}$$

The spectral curve X is given by

$$\begin{aligned} &F(z,w) = \mathrm{Det}\big(w\mathbb{1} - \mathbf{T}^0_{m;1,1}(z)\big) = w^2 - wf_1(z) + f_2(z) = 0 \\ &\text{where} \quad \deg f_1(z) = m \quad \deg f_2(z) = 2m-1 \end{aligned} \tag{3.2}$$

and its genus is $g = m-1$. The set $\{\mathbf{M}(z)\}_F$ (2.18) is obtained as the level set of

$$\mathbf{M}(z) = \mathbf{S}\mathbf{T}_{m;1,1}(z)\mathbf{S}^{-1} \qquad \mathbf{S} = \begin{pmatrix} \vec{e}_1 \\ \vec{e}_1\boldsymbol{\mu}_1 \end{pmatrix} \tag{3.3}$$

where $\mathbf{M}(z)$ has the form

$$\mathbf{M}(z) = \boldsymbol{\eta}_0 z^m + \cdots + \boldsymbol{\eta}_m \qquad \text{where} \quad \boldsymbol{\eta}_0 = \begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix} \quad \boldsymbol{\eta}_1 = \begin{pmatrix} 0 & 1 \\ * & * \end{pmatrix}$$

and other $\boldsymbol{\eta}_i$ are matrices without zero entries.

(ii) $\mathbf{T}_{m;1,2}(z)$. This is the case with the matrix

$$\mathbf{T}_{m;1,2}(z) = \boldsymbol{\mu}_-^{(1)} z^m + \boldsymbol{\mu}_1 z^{m-1} + \cdots + \boldsymbol{\mu}_{m-1} z + \boldsymbol{\mu}_+^{(2)} \qquad \text{for} \quad m \geqslant 2$$

and X is determined by

$$\begin{aligned} &F(z,w) = \mathrm{Det}\big(w\mathbb{1} - \mathbf{T}^0_{m;1,2}(z)\big) = w^2 - wzf_1'(z) + zf_2'(z) = 0 \\ &\text{where} \quad \deg f_1'(z) = m-1 \quad \deg f_2'(z) = 2m-2. \end{aligned} \tag{3.4}$$

The genus of X is $m-1$. By using the gauge matrix

$$\mathbf{S} = \begin{pmatrix} \vec{e}_2\boldsymbol{\mu}_-^{(1)} \\ \vec{e}_2 \end{pmatrix} \tag{3.5}$$

we obtain $\mathbf{M}(z)$ (2.19) with

$$\boldsymbol{\eta}_0 = \begin{pmatrix} * & 0 \\ 1 & 0 \end{pmatrix} \qquad \boldsymbol{\eta}_m = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$$

and the other $\boldsymbol{\eta}_i$ are the matrices with no zero entries.

One sees that both of $\{\mathbf{T}_{m;1,1}(z)\}_F$ and $\{\mathbf{T}_{m;1,2}(z)\}_F$ constitute the algebraic varieties of dimension m which is not equal to the genus of X. For example, by the definition (3.1) one sees that $\mathbf{T}_{m;1,1}(z)$ has $(4m+1)$ variables to which the fixed characteristic equation (3.2) gives $3m+1$ relations. Then we see $\{\mathbf{T}_{m;1,1}(z)\}_F$ constitutes the m-dimensional algebraic variety. The gauge matrix $\mathbf{S}$ reduces $\{\mathbf{T}_{m;1,1}(z)\}_F$ by one dimension, and $\{\mathbf{M}(z)\}_F$ becomes $(m-1)$ dimensional. After choosing the vector $\vec{a}(z) = (a_1(z), a_2(z))$ (2.20) the separation equation (2.22) is obtained as

$$B(z) = \begin{cases} T(z)_{1,2} = B_0\prod_{i=1}^{m-1}(z-z_i) & \vec{a}(z) = (1,0) \quad \text{for (i)} \\ T(z)_{2,1} = B_0 z\prod_{i=1}^{m-1}(z-z_i) & \vec{a}(z) = (0,1) \quad \text{for (ii)} \end{cases} \tag{3.6}$$

where $T(z)_{i,j} = \left(\mathbf{T}_{m;n_1,n_2}(z)\right)_{i,j}$. In both cases $B(z)$ generally has $m-1$ non-zero roots: $z_1, \ldots, z_{m-1}$, and each of them gives an eigenvalue

$$w_i = \begin{cases} T(z_i)_{2,2} & \text{for} \quad \text{(i)} \\ T(z_i)_{1,1} & \text{for} \quad \text{(ii)}. \end{cases}$$

In the level set $\left\{\mathbf{T}_{m;n_1,n_2}(z)\right\}_F$ the points (z_i, w_i) on X determine the effective divisor over X,

$$P = \sum_{i=1}^{m-1} [(z_i, w_i)] \in X(g) - D.$$

We remark that this divisor is invariant under the gauge transformation induced by $\mathbf{S}$, namely the solution of the separation equation does not change after replacing each $T(z)_{i,j}$ with a matrix element of $\mathbf{M}(z)$; $M(z)_{i,j}$. In this case X is linearly transformed to the hyperelliptic curve, and we can easily see the structure of D [9]. On the curve X, we have two infinity points $\infty_\pm$ and $m-1$ homomorphic 1-forms (2.26)

$$h_i(z, w) = z^{i-1} \qquad \text{for} \quad i = 1, \ldots, m-1.$$

Then D is written as [34]

$$D = \left\{ P = \sum_{i=1}^{m-1} [(z_i, w_i)] \mid z_i = z_j \text{for some } i \neq j, \text{ or } (z_i, w_i) = \infty_\pm \text{for some } i \right\}.$$

3.2. N = 3 case

The matrices (2.6) and (2.7) are written as

$$\boldsymbol{\mu}_-^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} \qquad \boldsymbol{\mu}_-^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & 0 \end{pmatrix} \qquad \boldsymbol{\mu}_-^{(0)} = \begin{pmatrix} * & * & 0 \\ * & * & * \\ * & * & * \end{pmatrix}$$

$$\boldsymbol{\mu}_+^{(1)} = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \qquad \boldsymbol{\mu}_+^{(2)} = \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \qquad \boldsymbol{\mu}_+^{(3)} = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{\mu}_+^{(0)} = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & * & * \end{pmatrix}.$$

We study six cases for $\mathbf{T}_{m;n_1,n_2}(z)$, $n_1 = 1, 2$ and $n_2 = 1, 2, 3$. For each $\mathbf{T}_{m;n_1,n_2}(z)$ we enumerate the forms of the spectral curve $F(z, w)$ given by $\mathbf{T}^0_{m;n_1,n_2}(z)$, the gauge matrix $\mathbf{S}$ and the matrix $\mathbf{M}(z)$ (2.18). In the following, unless a concrete form is shown, $\boldsymbol{\eta}_i$ (2.19) denote the matrices without zero entries.

(i) $\mathbf{T}_{m;1,1}(z)$. We start with the matrix

$$\mathbf{T}_{m;1,1}(z) = z^m \boldsymbol{\mu}_-^{(1)} + z^{m-1}\boldsymbol{\mu}_1 + \cdots + z\boldsymbol{\mu}_{m-1} + \boldsymbol{\mu}_+^{(1)}$$

whose characteristic polynomial is given by $\mathbf{T}^0_{m;1,1}(z)$ as

$$F(z, w) = w^3 - f_1(z)w^2 + f_2(z)w - f_3(z) \tag{3.7}$$

where $\deg f_1(z) = m$, $\deg f_2(z) = 2m$ and $\deg f_3(z) = 3m - 1$. The genus of the curve X is $g = 3m - 2$. The gauge matrix

$$\mathbf{S} = \begin{pmatrix} \vec{e}_1 \\ \vec{e}_1 \boldsymbol{\mu}_1 \boldsymbol{\mu}_-^{(1)} \\ \vec{e}_1 \boldsymbol{\mu}_1 \end{pmatrix}$$

introduces (2.19) of the form

$$\mathbf{M}(z) = z^m \begin{pmatrix} 0 & 0 & 0 \\ * & * & * \\ 0 & 1 & 0 \end{pmatrix} + z^{m-1} \begin{pmatrix} 0 & 0 & 1 \\ * & * & * \\ * & * & * \end{pmatrix} + O(z^{m-2}).$$

(ii) $\mathbf{T}_{m;2,2}(z)$. The matrix is

$$\mathbf{T}_{m;2,2}(z) = z^m \boldsymbol{\mu}_-^{(2)} + z^{m-1}\boldsymbol{\mu}_-^{(0)} + z^{m-2}\boldsymbol{\mu}_2 + \cdots + z\boldsymbol{\mu}_{m-1} + \boldsymbol{\mu}_+^{(2)} \qquad \text{for} \quad m \geqslant 2$$

and X is given by $\mathbf{T}^0_{m;2,2}(z)$,

$$F(z, w) = w^3 - zf_1'(z)w^2 + zf_2'(z)w - zf_3'(z) \tag{3.8}$$

where $\deg f_1'(z) = m - 2$, $\deg f_2'(z) = 2m - 2$ and $\deg f_3'(z) = 3m - 3$. The genus is $g = 3m - 3$. With the gauge matrix

$$\mathbf{S} = \begin{pmatrix} \vec{e}_3\boldsymbol{\mu}_-^{(2)} \\ \vec{e}_3\boldsymbol{\mu}_-^{(2)}\boldsymbol{\mu}_+^{(2)} \\ \vec{e}_3 \end{pmatrix}$$

(2.19) is obtained as

$$\mathbf{M}(z) = z^m \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}.$$

(iii) $\mathbf{T}_{m;1,3}(z)$. For the matrix

$$\mathbf{T}_{m;1,3}(z) = z^m \boldsymbol{\mu}_-^{(1)} + z^{m-1}\boldsymbol{\mu}_1 + \cdots + z\boldsymbol{\mu}_+^{(0)} + \boldsymbol{\mu}_+^{(3)} \qquad \text{for} \quad m \geqslant 2$$

X is given by

$$F(z, w) = w^3 - zf_1'(z)w^2 + z^2 f_2'(z)w - z^2 f_3'(z) \tag{3.9}$$

where $\deg f_1'(z) = m - 1$, $\deg f_2'(z) = 2m - 2$ and $\deg f_3'(z) = 3m - 3$. The genus of X is $g = 3m - 3$. The gauge

$$\mathbf{S} = \begin{pmatrix} \vec{e}_1 \\ \vec{e}_1\boldsymbol{\mu}_+^{(3)}\boldsymbol{\mu}_+^{(0)} \\ \vec{e}_1\boldsymbol{\mu}_+^{(3)} \end{pmatrix}$$

gives (2.19),

$$\mathbf{M}(z) = z^m \begin{pmatrix} 0 & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix} + \cdots + z \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(iv) $\mathbf{T}_{m;2,1}(z)$. The matrix is

$$\mathbf{T}_{m;2,1}(z) = z^m \boldsymbol{\mu}_-^{(2)} + z^{m-1}\boldsymbol{\mu}_-^{(0)} + z^{m-2}\boldsymbol{\mu}_2 + \cdots + z\boldsymbol{\mu}_{m-1} + \boldsymbol{\mu}_+^{(1)} \qquad \text{for} \quad m \geqslant 2$$

and X is given by (3.7) with $\deg f_1(z) = m - 1$, $\deg f_2(z) = 2m - 1$ and $\deg f_3(z) = 3m - 2$. The genus of X is $g = 3m - 3$. The gauge matrix and the matrix (2.19) are

$$\mathbf{S} = \begin{pmatrix} \vec{e}_3\boldsymbol{\mu}_-^{(2)} \\ \vec{e}_3\boldsymbol{\mu}_-^{(2)}\boldsymbol{\mu}_+^{(1)} \\ \vec{e}_3 \end{pmatrix} \qquad \mathbf{M}(z) = z^m \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & 1 & 0 \\ * & * & * \\ 0 & 0 & * \end{pmatrix}.$$

(v) $\mathbf{T}_{m;1,2}(z)$. The matrix

$$\mathbf{T}_{m;1,2}(z) = z^m \boldsymbol{\mu}_-^{(1)} + z^{m-1}\boldsymbol{\mu}_1 + \cdots + z\boldsymbol{\mu}_{m-1} + \boldsymbol{\mu}_+^{(2)}$$

has the spectral curve (3.8) with $\deg f_1'(z) = m-1, \deg f_2'(z) = 2m-1$ and $\deg f_3'(z) = 3m-2$, whose genus is $3m-2$. The gauge matrix and the matrix (2.19) are

$$\mathbf{S} = \begin{pmatrix} \vec{e}_1 \\ \vec{e}_1\big(\boldsymbol{\mu}_+^{(2)}\big)^2 \\ \vec{e}_1\boldsymbol{\mu}_+^{(2)} \end{pmatrix} \qquad \mathbf{M}(z) = z^m \begin{pmatrix} 0 & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix} + \cdots + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

(vi) $\mathbf{T}_{m;2,3}(z)$. When $m \geqslant 3$, the matrix is defined as

$$\mathbf{T}_{m;2,3}(z) = z^m \boldsymbol{\mu}_-^{(2)} + z^{m-1}\boldsymbol{\mu}_-^{(0)} + z^{m-2}\boldsymbol{\mu}_2 + \cdots + z^2\boldsymbol{\mu}_{m-2} + z\boldsymbol{\mu}_+^{(0)} + \boldsymbol{\mu}_+^{(3)}.$$

Its spectral curve is given by (3.9) with $\deg f_1'(z) = m-2, \deg f_2'(z) = 2m-3$ and $\deg f_3'(z) = 3m-4$, and the genus is $3m-4$. The gauge matrix and the matrix (2.19) are obtained as

$$\mathbf{S} = \begin{pmatrix} \vec{e}_3\boldsymbol{\mu}_+^{(0)} \\ \vec{e}_3\boldsymbol{\mu}_+^{(0)}\boldsymbol{\mu}_-^{(2)} \\ \vec{e}_3 \end{pmatrix} \qquad \mathbf{M}(z) = z^m \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & * & 0 \end{pmatrix} + \cdots + z \begin{pmatrix} * & * & * \\ * & * & * \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.10}$$

For the case of $m = 2$, we have

$$\mathbf{T}_{2;2,3}(z) = z^2\boldsymbol{\mu}_-^{(2)} + z\big(\boldsymbol{\mu}_-^{(0)} \cap \boldsymbol{\mu}_+^{(0)}\big) + \boldsymbol{\mu}_+^{(3)} \qquad \text{where} \quad \big(\boldsymbol{\mu}_-^{(0)} \cap \boldsymbol{\mu}_+^{(0)}\big) = \begin{pmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{pmatrix}.$$

Following this form, the gauge becomes

$$\mathbf{S} = \begin{pmatrix} \vec{e}_3\big(\boldsymbol{\mu}_-^{(0)} \cap \boldsymbol{\mu}_+^{(0)}\big) \\ \vec{e}_3\big(\boldsymbol{\mu}_-^{(0)} \cap \boldsymbol{\mu}_+^{(0)}\big)\boldsymbol{\mu}_-^{(2)} \\ \vec{e}_3 \end{pmatrix}.$$

The associated characteristic polynomial and the matrix (2.19) are obtained by substituting $m = 2$ in (3.9) and (3.10).

We construct the set of representatives $\{\mathbf{M}(z)\}_F$ (2.18) based on $F(z, w)$ and $\mathbf{M}(z)$ for each case. One sees that in all cases the gauge matrices $\mathbf{S}$ reduce the dimension of the variety $\big\{\mathbf{T}_{m;n_1,n_2}(z)\big\}_F$ by two. The separation equation differs depending on which of $\vec{e}_1$ and $\vec{e}_3$ the gauge matrix $\mathbf{S}$ includes. For the cases of (i), (iii) and (v), we have the invariant separation equation (2.22)

$$B(z) = \mathrm{Det}\begin{pmatrix} (T(z)_{1,2}, T(z)_{1,3}) \\ (T(z)_{1,2}, T(z)_{1,3})\begin{pmatrix} T(z)_{2,2} & T(z)_{2,3} \\ T(z)_{3,2} & T(z)_{3,3} \end{pmatrix} \end{pmatrix} = 0 \qquad \text{where} \quad \vec{a} = (1, 0, 0)$$

and for the rest of the cases,

$$B(z) = \mathrm{Det}\begin{pmatrix} (T(z)_{3,1}, T(z)_{3,2}) \\ (T(z)_{3,1}, T(z)_{3,2})\begin{pmatrix} T(z)_{1,1} & T(z)_{1,2} \\ T(z)_{2,1} & T(z)_{2,2} \end{pmatrix} \end{pmatrix} = 0 \qquad \text{where} \quad \vec{a} = (0, 0, 1).$$

In all cases $B(z)$ has a form as (2.23), and each of them gives the eigenvalue by

$$w_i = \begin{cases} \mathrm{Det}\begin{pmatrix} T(z_i)_{1,2} & T(z_i)_{1,3} \\ T(z_i)_{3,2} & T(z_i)_{3,3} \end{pmatrix} \Big/ T(z_i)_{1,2} & \text{for (i), (iii), (v)} \\ \mathrm{Det}\begin{pmatrix} T(z_i)_{1,1} & T(z_i)_{1,2} \\ T(z_i)_{3,1} & T(z_i)_{3,2} \end{pmatrix} \Big/ T(z_i)_{3,2} & \text{for (ii), (iv), (vi).} \end{cases}$$

In conclusion, the separation equation uniquely determines the effective divisor $P \in X(g) - D$ which is invariant under the gauge $\mathbf{S}$.

3.3. General N cases

In the case of general $N \geqslant 4$, we have $N(N-1)$ kinds of monodromy matrices $\mathbf{T}_{m;n_1,n_2}(z)$. When $n_1 = n_2 = 1$, the spectral curve X is given by (2.1) where $\deg f_i(z) = im$, for $i = 1, \ldots, N-1$, and $f_N(z) = Nm - 1$. Then the genus is $g = \frac{1}{2}(N-1)(Nm-2)$. For each $\mathbf{T}_{m;1,1}(z)$ we have a gauge matrix [15];

$$\mathbf{S} = \begin{pmatrix} \vec{e}_1 \\ \vec{e}_1 \boldsymbol{\mu}_1 (\boldsymbol{\mu}_-^{(1)})^{N-2} \\ \vdots \\ \vec{e}_1 \boldsymbol{\mu}_1 \boldsymbol{\mu}_-^{(1)} \\ \vec{e}_1 \boldsymbol{\mu}_1 \end{pmatrix} \tag{3.11}$$

which reduces the variety of $\{\mathbf{T}_{m;1,1}(z)\}_F$ by $N-1$ dimensions. Using the elements of $\mathbf{T}_{m;1,1}(z)$ given by

$$\mathbf{T}_{m;1,1}(z) = \begin{pmatrix} a(z) & \vec{b}(z) \\ \vec{c}(z)^T & \mathbf{d}(z) \end{pmatrix}$$

the separation equation is defined as [31, 32]

$$B(z) \equiv \mathrm{Det} \begin{pmatrix} \vec{b}(z) \\ \vec{b}(z)\,\mathbf{d}(z) \\ \vec{b}(z)\,\mathbf{d}(z)^2 \\ \vdots \\ \vec{b}(z)\,\mathbf{d}(z)^{N-2} \end{pmatrix}.$$

Then $B(z)$ becomes a polynomial of z of degree g, and the zeros of $B(z)$ is invariant under the gauge transformation induced by $\mathbf{S}$ [32].

Instead of showing other cases, based on the above concrete studies, we introduce the conjecture for $\mathbf{S}$ as follows:

Conjecture 3.1. *For $\mathbf{T}_{m;n_1,n_2}(z)$ (2.10) there is a gauge matrix $\mathbf{S}$ (2.18) of the form*

$$\mathbf{S} = \begin{pmatrix} \vec{e}_1 \\ \vec{e}_1 \boldsymbol{\mu}_a \boldsymbol{\mu}_b^{N-2} \\ \vdots \\ \vec{e}_1 \boldsymbol{\mu}_a \boldsymbol{\mu}_b \\ \vec{e}_1 \boldsymbol{\mu}_a \end{pmatrix} \quad \text{for even } L \qquad \begin{pmatrix} \vec{e}_N \boldsymbol{\mu}_a \\ \vec{e}_N \boldsymbol{\mu}_a \boldsymbol{\mu}_b \\ \vdots \\ \vec{e}_N \boldsymbol{\mu}_a \boldsymbol{\mu}_b^{N-2} \\ \vec{e}_N \end{pmatrix} \quad \text{for odd } L \tag{3.12}$$

where $\boldsymbol{\mu}_a, \boldsymbol{\mu}_b \in \left\{ \boldsymbol{\mu}_-^{(n_1)}, \boldsymbol{\mu}_-^{(n_1-N+1)}, \boldsymbol{\mu}_+^{(n_2)}, \boldsymbol{\mu}_+^{(n_2-N)}, \boldsymbol{\mu}_j, j = 2, \ldots, m-2 \right\}$, such that $\mathbf{S}$ reduces $\{\mathbf{T}_{m;n_1,n_2}(z)\}_F$ to a g-dimensional variety $\{\mathbf{M}(z)\}_F$ (2.18) and that the associated separation equation (2.22) of the form (2.23) has zeros invariant under $\mathbf{S}$.

We briefly remark on the diagram (1.2). If we get the gauge matrix $\mathbf{S}$ which solves problem 2.6 then (a) becomes surjective, since $\{\mathbf{M}(z)\}_F$ is the set of representatives. Therefore there exists a map (c) which makes the diagram (1.2) commutative.

4. Integrability of LV(N, L)

4.1. Spectral curve and Poisson structure for LV(N, L)

We introduce the N by N Lax matrix for the extended Lotka–Volterra lattice (1.3) as

$$\tilde{\mathbf{L}}_n(z) = \frac{1}{z V_n^{\frac{N-1}{N}}} \left(\sum_{k=1}^{N-1} V_n \mathbf{E}_{k,k+1} + z^N (-1)^{N-1} \mathbf{E}_{N,1} + z^N (-1)^{N-2} \mathbf{E}_{N,2} \right). \tag{4.1}$$

We have modified the original Lax matrix [17], and (4.1) comes from $\overline{\mathbf{L}}_n(z)$ in [26]. Note that $\tilde{\mathbf{L}}_n(z)$ has been normalized so that $\mathrm{Det}\,\tilde{\mathbf{L}}_n(z) = 1$. The monodromy matrix $\tilde{\mathbf{T}}(z)$ of an L-periodic model LV(N, L) is defined as

$$\tilde{\mathbf{T}}(z) = \prod_{k=1}^{\overset{L}{\curvearrowleft}} \tilde{\mathbf{L}}_k(z). \tag{4.2}$$

The characteristic equation of $\tilde{\mathbf{T}}(z)$,

$$\mathrm{Det}(w\mathbb{1} - \tilde{\mathbf{T}}(z)) = 0 \tag{4.3}$$

gives an algebraic curve $\tilde{X}$. For this equation we have the automorphism τ of order N,

$$\tau : (z, w) \mapsto (\epsilon z, \epsilon^{-k_2} w)$$

where $\epsilon = \mathrm{e}^{\frac{2\pi \mathrm{i}}{N}}$, and k_2 is defined by (2.15). We define the matrix $\mathbf{T}_{\mathrm{LV}}(z)$,

$$\mathbf{T}_{\mathrm{LV}}(z) \equiv z^{\frac{k_2}{N}} \tilde{\mathbf{T}}\left(z^{\frac{1}{N}}\right) \tag{4.4}$$

then its matrix elements become polynomials of z and $\mathrm{Det}\,\mathbf{T}_{\mathrm{LV}}(z) = z^{k_2}$. The characteristic equation of $\mathbf{T}_{\mathrm{LV}}(z)$ gives the quotient curve $\tilde{X}/\tau$.

On the other hand, the Hamiltonian structure of LV(N, L) is defined by the Poisson brackets [17]

$$\{V_n, V_m\} = 2 V_n V_m \sum_{k=1}^{N-1} (\delta_{m,n+k} - \delta_{m,n-k}) \tag{4.5}$$

and the Hamiltonian $H_1 = \sum_{n=1}^{L} V_n$. Using these settings, the time evolution (1.3) is given by

$$\frac{\partial V_n}{\partial t_1} = \{V_n, H_1\}$$

with $t = t_1$. We let $\mathcal{A}_{\mathrm{LV}}$ be the Poisson bracket algebra for $\mathbb{C}[V_n; n \in \mathbb{Z}/L\mathbb{Z}]$ whose defining relations are given by (4.5). We have the centre of $\mathcal{A}_{\mathrm{LV}}$ denoted by $\mathcal{A}_{\mathrm{LV}}^0$ as follows:

Proposition 4.1. *The centre $\mathcal{A}_{\mathrm{LV}}^0$ is generated by the variables*

$$\mathcal{P}_k^{(i)} = \prod_{n=0}^{\frac{L}{k}-1} (V_{kn+i}) \qquad \textit{for} \quad k \in \mathcal{K} \quad i \in \{1, \ldots, k\} \tag{4.6}$$

where

$$\mathcal{K} = \{k \in \{1, \ldots, N\} \mid k|N \text{ or } k|(N-1)\} \sqcap \{k \mid k|L\}. \tag{4.7}$$

Here $k|L$ means that L is a multiple of k.

See appendix B for the proof. Since the set $\{\mathcal{P}_k^{(i)} | i \in \{1, \ldots, k\}\}$ is generated by $\{\mathcal{P}_{k'}^{(j)} | j \in \{1, \ldots, k'\}\}$ for $k|k'$, to generate $\mathcal{A}_{\mathrm{LV}}^0$ it is enough to have a set

$$\{\mathcal{P}_k^{(i)} \mid k \in \mathcal{K}_0, i \in \{1, \ldots, k\}\}$$

where $\mathcal{K}_0 = \{\max[k \in \mathcal{K} \text{ for } k|N], \max[k \in \mathcal{K} \text{ for } k|(N-1)]\}$. Then the number of independent generators of $\mathcal{A}_{\mathrm{LV}}^0$ is

$$n_0 = \sum_{k \in \mathcal{K}_0} k - (|\mathcal{K}_0| - 1). \tag{4.8}$$

Based on the structure of the monodromy matrix (4.2), we introduce a variable

$$\mathcal{P}_0 \equiv \prod_{n=1}^{L} (V_n)^{-\frac{1}{N}} = \left(\mathcal{P}_1^{(1)}\right)^{-\frac{1}{N}}$$

that Poisson commutes with every V_n. Therefore $\mathcal{A}_{\mathrm{LV}}$ is naturally extended to the Poisson bracket algebra over $\mathbb{C}(\mathcal{P}_0, V_n; n \in \mathbb{Z}/L\mathbb{Z})$. We denote this algebra by $\mathcal{A}'_{\mathrm{LV}}$.

A family of the integrals of motion (IM) for LV(N, L) which includes the Hamiltonian H_1 appears as coefficients of the characteristic equation (4.3).

Proposition 4.2 [19, 35]**.** *The IM is a commuting subalgebra of $\mathcal{A}'_{\mathrm{LV}}$.*

Proof. We show the outline of the proof. We introduce the variable transformation

$$V_n = (P_n P_{n+1} \cdots P_{n+N-1})^{-1} Q_n^{-1} Q_{n+N-1}$$

where P_n, Q_n are canonical variables,

$$\{P_n, Q_m\} = \delta_{n,m} P_n Q_n \qquad \{P_n, P_m\} = \{Q_n, Q_m\} = 0. \tag{4.9}$$

Then the matrix $\mathbf{T}_{\mathrm{LV}}(z)$ is transformed into $\mathbf{T}_C(z)$ by using a diagonal matrix $\mathbf{B}_1 = \mathbf{B}_1(P_1, \ldots, P_{N-1}, Q_1, \ldots, Q_{N-1})$

$$\mathbf{T}_C(z) = \mathbf{B}_1 \mathbf{T}_{\mathrm{LV}}(z) (\mathbf{B}_1)^{-1} \tag{4.10}$$

which satisfies the fundamental Poisson relation (2.17)

$$\{\mathbf{T}_C(z) \overset{\otimes}{,} \mathbf{T}_C(z')\} = [\mathbf{r}(z/z'), \mathbf{T}_C(z) \otimes \mathbf{T}_C(z')] \tag{4.11}$$

with the r-matrix (2.13). See [26, 35] for details of the matrices $\mathbf{B}_1$ and $\mathbf{T}_C(z)$. Note that the characteristic equation for the matrix $\mathbf{T}_C(z)$ is obtained from (4.3) by a transformation $(z, w) \mapsto \left(z^{\frac{1}{N}}, w z^{\frac{k_2}{N}}\right)$, and that the coefficients of the characteristic polynomial belong to $\mathbb{C}[\mathcal{P}_0, V_n; n \in \mathbb{Z}]$. Then the proposition follows. □

We introduce a grading on $\mathcal{A}_{\mathrm{LV}}$ as $\deg V_n = 1$. Since the IM are obtained as homogeneous polynomials of V_n, we can identify each of IM based on the grading. For instance, for the Hamiltonian H_1 we have $\deg H_1 = 1$. Let n_H be the number of the elements of IM in $\mathcal{A}_{\mathrm{LV}}$. By putting the IM in the order of the grading, we obtain

$$H_1, H_2, \ldots, H_{n_H}. \tag{4.12}$$

The proposition 4.2 yields

Corollary 4.3. *The family of IM generates n_H commuting flows for LV(N, L) defined as*

$$\frac{\partial \mathcal{O}}{\partial t_i} \equiv \{\mathcal{O}, H_i\} \qquad \textit{for} \quad \mathcal{O} \in \mathcal{A}'_{\mathrm{LV}} \quad i = 1, \ldots, n_H. \tag{4.13}$$

We comment that in [18] the Hamiltonian structure of $\mathrm{LV}(N, L)$ is studied by applying the r-matrix method [5] to the *big* L by L Lax matrix, and the involution of IM is clarified by this approach. Since our aim here is to establish the eigenvector map for $\mathrm{LV}(N, L)$ based on the monodromy matrix (2.17), it is important to get the *small* N by N monodromy matrix with the fundamental Poisson relation (4.11).

4.2. Realization of $\mathbf{M}_F(z)$ and the integrable structure of $LV(N, L)$

We find that the matrix $\mathbf{T}_C(z)$ (4.10) gives a realization of $\mathbf{T}_{m;n_1,n_2}(z)$ (2.10), namely both the form and the Poisson structure of $\mathbf{T}_C(z)$ coincide with those of $\mathbf{T}_{m;n_1,n_2}(z)$. We obtain a similar relation as (2.16) as follows:

Proposition 4.4. *Under a condition*

$$\mathrm{Det}\,\mathbf{T}_{m;n_1,n_2}(z) = z^{n_2-1} \tag{4.14}$$

$\mathbf{T}_C(z)$ *realizes* $\mathbf{T}_{m;n_1,n_2}(z)$ *and they are related as*

$$\mathbf{T}_C(z) = \begin{cases} \mathbf{T}_{m;1,1}(z) & \textit{for} \quad k_1 = k_2 = 0 \\ \mathbf{T}_{m+1;N-k_1,k_2+1}(z) & \textit{for} \quad k_1, k_2 \neq 0 \quad 0 \leqslant k_1 - k_2 \leqslant N-2 \\ \mathbf{T}_{m+2;N-k_1,k_2+1}(z) & \textit{for} \quad k_1 - k_2 \leqslant -1. \end{cases} \tag{4.15}$$

Proof. First we check the coincidence of the form. Note that the condition (4.14) comes from the normalization of $\tilde{\mathbf{L}}_n(z)$. The Lax matrices $z^{\frac{1}{N}}\tilde{\mathbf{L}}_n(z^{\frac{1}{N}})$ (4.1) and $\mathbf{L}_n(z)$ (2.11) have the same form as polynomial matrices. Then we see that $\mathbf{T}_{\mathrm{LV}}(z)$ (4.4) and $z^{-\frac{L}{N}+\frac{k_2}{N}+m_2}\mathbf{T}^{(L)}(z)$ (A.1) have the same form. Since the gauge $\mathbf{B}_1$ (4.10) is diagonal and does not change the form of $\mathbf{T}_{\mathrm{LV}}(z)$, we obtain the correspondence of $\mathbf{T}_C(z) = \mathbf{B}_1\mathbf{T}_{\mathrm{LV}}(z)\mathbf{B}_1^{-1}$ and $\mathbf{T}^{(L)}(z)$. By using lemma 2.3 and the relation $L = Nm_2 + k_2$ (2.15), we obtain (4.15).

Next, we observe the Poisson structure. The conditions (4.14) and (2.17) do not contradict each other, since proposition 2.5 says that $\mathrm{Det}\,\mathbf{T}_{m;n_1,n_2}(z)$ belongs to the centre of $\mathcal{A}_C$. Then from (2.17) and (4.11), the monodromy matrices $\mathbf{T}_C(z)$ obviously have the same Poisson structure as that of $\mathbf{T}_{m;n_1,n_2}(z)$. □

Once we associate $\mathbf{T}_C(z)$ with $\mathbf{T}_{m;n_1,n_2}(z)$, $\mathbf{T}_C(z)$ realizes $\left\{\mathbf{T}_{m;n_1,n_2}(z)\right\}_F$ where $\mathbf{T}^0_{m;n_1,n_2}(z)$ corresponds to the initial condition for $\mathbf{T}_C(z)$. We also see $\tilde{X}/\tau \simeq X$.

In the following we discuss the integrability of $\mathrm{LV}(N, L)$ based on the representative $\{\mathbf{M}(z)\}_F$ (2.18) and the Poisson bracket algebra $\mathcal{A}_M$ realized by $\mathrm{LV}(N, L)$. We introduce an important proposition:

Proposition 4.5. *If the gauge matrix* $\mathbf{S}$ *which meets the conditions in conjecture 3.1 exists, then (i)* $\mathcal{A}_M \subset \mathcal{A}'_{\mathrm{LV}'}$ *(ii) the separation equation (2.22) gives* g *algebraic relations between* $z_i (i = 1, \ldots, g)$ *and* $V_n (n \in \mathbb{Z}/L\mathbb{Z})$.

Recall that the matrix $\mathbf{T}_C(z)$ is no longer written in terms of the dynamical variables of $\mathrm{LV}(N, L)$, but of the canonical variables (4.9). Therefore $\mathcal{A}_M \subset \mathcal{A}_C$ is trivial but the claim (i) in the above proposition is not. This claim was conjectured in [26] and now is proved in a simple way. We add the proof of proposition 4.5 in appendix C.

On the tangent space of $\boldsymbol{\mathcal{M}}_F$ there is the g-dimensional invariant vector field which induces the evolution of the divisor P (2.24) linearized on $J_{\mathrm{aff}}(X)$. When n_H is equal to g, we can identify the coordinates on $J_{\mathrm{aff}}(X)$ with the times t_i (4.13), and get z_i as a functions of t_i; $z_i = z_i(t_1, \ldots, t_g)$. Further, if $n_H = \frac{1}{2}(L - n_0)$ is satisfied, we can reduce the integrability of $\mathrm{LV}(N, L)$ to L independent algebraic relations between the dynamical variables of $\mathrm{LV}(N, L)$

and H_i (4.12), z_i (2.22) and n_0 generators of $\mathcal{A}_{\mathrm{LV}}^0$ (4.8). We summarize the integrability of $\mathrm{LV}(N, L)$ as follows:

Proposition 4.6. *$LV(N, L)$ is algebraic completely integrable if*

$$g = n_H = \tfrac{1}{2}(L - n_0) \tag{4.16}$$

and proposition 4.5 is satisfied.

In section 3, we solved problem 2.6 for the cases of $N = 2, 3$ and the special case of general N. We obtained the gauge matrices $\mathbf{S}$ (2.18) which satisfy conjecture 3.1, then proposition 4.5 is satisfied for these cases. The last case corresponds to $\mathrm{LV}(N, L)$ with the special periodicity $L = N(N-1)m$ studied in [26] where (4.16) was proved and proposition 4.5 was conjectured. Having proved proposition 4.5, we can now conclude that

Theorem 4.7. *$LV(N, N(N-1)m)$ is algebraic completely integrable.*

In the following, we investigate propositions 4.5 and 4.6 for the results in section 3 and show theorem 1.1.

4.3. LV(2, L)

Depending on the periodicity L we have two cases:

(i) $L = 2m$, $\mathbf{T}_C(z) = \mathbf{T}_{m;1,1}(z)$. The IM are obtained as the coefficients of (3.2) with

$$f_1(z) = \mathcal{P}_0\big(z^m + H_1 z^{m-1} + H_2 z^{m-2} - \cdots + zH_{m-1} + \big(\mathcal{P}_2^{(1)} + \mathcal{P}_2^{(2)}\big)\big).$$

Here we have $m-1$ independent IM identified by their degree, $\deg H_i = i$. The centre $\mathcal{A}_{\mathrm{LV}}^0$ is generated by two of $\mathcal{P}_1^{(1)}$, $\mathcal{P}_2^{(1)}$ and $\mathcal{P}_2^{(2)}$. The genus of X is equal to n_H.

(ii) $L = 2m+1$, $\mathbf{T}_C(z) = \mathbf{T}_{m+1;1,2}(z)$. We have m independent IM given by (3.4) with

$$f_1'(z) = \mathcal{P}_0(z^m - H_1 z^{m-1} + H_2 z^{m-2} - \cdots + (-)^m H_m)$$

where $\deg H_i = i$. The centre $\mathcal{A}_{\mathrm{LV}}^0$ is generated by $\mathcal{P}_1^{(1)}$ only.

In both cases (4.16) is satisfied and the gauge matrices $\mathbf{S}$ (3.3) and (3.5) fulfil proposition 4.5. Therefore we conclude that $\mathrm{LV}(2, L)$ is algebraic completely integrable. The correspondence of the periodicity L and the genus g is summarized as

L	3	4	5	6	7	8	...	$2m$	$2m+1$	...
g	1	1	2	2	3	3	...	$m-1$	m	...

.

4.4. LV(3, L)

The periodicity L is classified into six cases.

(i) $L = 6m$, $\mathbf{T}_C(z) = \mathbf{T}_{m;1,1}(z)$. The IM are obtained as

$$f_1(z) = \mathcal{P}_0^2(f_{3m} z^m + f_{3m+1} z^{m-1} + \cdots + f_{4m})$$
$$f_2(z) = \mathcal{P}_0(z^{2m} + f_1 z^{2m-1} + \cdots + f_{2m})$$

where we set f_i so as to accomplish $\deg f_i = i$. The generators of $\mathcal{A}_{\mathrm{LV}}^0$ have the ordering as $\deg \mathcal{P}_2^{(i)} = 3m$, $\deg \mathcal{P}_3^{(i)} = 2m$, then f_{3m}, f_{4m} and f_{2m} belong to $\mathcal{A}_{\mathrm{LV}}^0$. Actually, we have relations

$$z^2 + f_{3m} z + \mathcal{P}_1^{(1)} = \big(z - \mathcal{P}_2^{(1)}\big)\big(z - \mathcal{P}_2^{(2)}\big) \tag{4.17}$$

$$z^3 + z^2 f_{2m} + z f_{4m} + \mathcal{P}_1^{(1)} = \big(z - \mathcal{P}_3^{(1)}\big)\big(z - \mathcal{P}_3^{(2)}\big)\big(z - \mathcal{P}_3^{(1)}\big). \tag{4.18}$$

In conclusion we have $n_H = 3m - 2$ which is equal to g, and $n_0 = 4$.

(ii) $L = 6m + 1$, $\mathbf{T}_C(z) = \mathbf{T}_{m+1;2,2}(z)$. We have

$$f_1'(z) = \mathcal{P}_0^2(f_{3m+1}z^{m-1} + f_{3m+2}z^{m-2} + \cdots + f_{4m})$$
$$f_2'(z) = \mathcal{P}_0(z^{2m} + f_1 z^{2m-1} + \cdots + f_{2m}).$$

In this case we have only a generator of $\mathcal{A}^0_{\mathrm{LV}}$; $\mathcal{P}_1^{(1)}$, and no f_i belongs to $\mathcal{A}^0_{\mathrm{LV}}$. Then $n_H = 3m$ and $n_0 = 1$.

(iii) $L = 6m + 2$, $\mathbf{T}_C(z) = \mathbf{T}_{m+1;1,3}(z)$.

$$f_1'(z) = \mathcal{P}_0^2(f_{3m+1}z^{m-1} + f_{3m+2}z^{m-2} + \cdots + f_{4m+1})$$
$$f_2'(z) = \mathcal{P}_0(z^{2m} + f_1 z^{2m-1} + \cdots + f_{2m}).$$

Since $\deg \mathcal{P}_2^{(i)} = 3m + 1$, we see $f_{3m+1} \in \mathcal{A}^0_{\mathrm{LV}}$, which satisfies a relation similar to (4.17). Then we have $n_H = 3m$ and $n_0 = 2$.

(iv) $L = 6m + 3$, $\mathbf{T}_C(z) = \mathbf{T}_{m+1;2,1}(z)$.

$$f_1(z) = \mathcal{P}_0^2(f_{3m+2}z^{m} + f_{3m+2}z^{m-1} + \cdots + f_{4m+2})$$
$$f_2(z) = \mathcal{P}_0(z^{2m+1} + f_1 z^{2m} + \cdots + f_{2m+1}).$$

Since $\deg \mathcal{P}_2^{(i)} = 2m + 1$, we see $f_{4m+2}, f_{2m+1} \in \mathcal{A}^0_{\mathrm{LV}}$, which satisfy a relation similar to (4.18). Then we have $n_H = 3m$ and $n_0 = 3$.

The remaining cases,

(v) $L = 6m + 4$, $\mathbf{T}_C(z) = \mathbf{T}_{m+1;1,2}(z)$

(vi) $L = 6m + 5$, $\mathbf{T}_C(z) = \mathbf{T}_{m+2;2,3}(z)$

permit the same analysis.

For all L we have n_H and n_0 which satisfy (4.16). Recall that in section 3.2 we have constructed the $\{\mathbf{M}(z)\}_F$ with the gauge matrices $\mathbf{S}$ which meet proposition 4.5. Herewith we prove the algebraic complete integrability of LV$(3, L)$. As same as the $N = 2$ case, we summarize the correspondence of L and g:

L	5	6	7	8	9	10	…	$6m$	$6m+1$	$6m+2$	$6m+3$	$6m+4$	$6m+5$
g	2	1	3	3	3	4	…	$3m-2$	$3m$	$3m$	$3m$	$3m+1$	$3m+2$

.

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Appendix A

Proof of lemma 2.3. We show the outline of the proof. We use the integers defined at (2.14) and (2.15), and set a matrix $\mathbf{T}^{(L)}(z)$,

$$\mathbf{T}^{(L)}(z) = z^{-m_2}\prod_{n=1}^{L}\mathbf{L}_n(z). \tag{A.1}$$

By definition, first we have

$$\mathbf{T}^{(1)}(z) = \mathbf{L}_1(z) = \boldsymbol{\mu}_-^{(N-1)} z + \left(\boldsymbol{\mu}_-^{(0)} \cap \boldsymbol{\mu}_+^{(2)}\right).$$

Therefore we obtain the correspondence $\mathbf{T}^{(1)}(z) = \mathbf{T}_{1;N-1,2}$. Assume $\mathbf{T}^{(L)}(z) = \mathbf{T}_{m;n_1,n_2}(z)$. When we set $\mathbf{T}^{(L)}(z) = \left(t_{i,j}^{(L)}\right)_{1 \leqslant i,j \leqslant N}$, $\mathbf{T}^{(L+1)}(z)$ are related to $\mathbf{T}^{(L)}(z)$ as

$$\mathbf{T}^{(L+1)}(z) = \begin{cases} \sum_{j=1}^{N} \left(\sum_{i=1}^{N-1} \mathbf{E}_{i,j} l_i^{(L+1)} t_{i+1,j}^{(L)} + z \mathbf{E}_{N,j} \left(l_N^{(L+1)} t_{1,j}^{(L)} + l_0^{(L+1)} t_{2,j}^{(L)} \right) \right) & \text{for} \quad n_2 \neq N \\ \sum_{j=1}^{N} \left(\frac{1}{z} \sum_{i=1}^{N-1} \mathbf{E}_{i,j} l_i^{(L+1)} t_{i+1,j}^{(L)} + \mathbf{E}_{N,j} \left(l_N^{(L+1)} t_{1,j}^{(L)} + l_0^{(L+1)} t_{2,j}^{(L)} \right) \right) & \text{for} \quad n_2 = N \end{cases}$$

then we find the correspondence

$$\mathbf{T}^{(L+1)}(z) = \begin{cases} \mathbf{T}_{m;n_1-1,n_2+1}(z) & \text{for} \quad n_1 \neq 1, n_2 \neq N \\ \mathbf{T}_{m+1;N-1,n_2+1}(z) & \text{for} \quad n_1 = 1, n_2 \neq N \\ \mathbf{T}_{m;N-1,1}(z) & \text{for} \quad n_1 = 1, n_2 = N \\ \mathbf{T}_{m-1;n_1-1,1}(z) & \text{for} \quad n_1 \neq 1, n_2 = N. \end{cases}$$

By induction, we obtain (2.16). □

Appendix B

Proof of proposition 4.1. Based on the periodicity L and the Poisson relations (4.5), we can set candidates for the generators of $\mathcal{A}_{\mathrm{LV}}^0$ as

$$\mathcal{P}_k^{(i)} = \prod_{n=0}^{\frac{L}{k}-1} (V_{kn+i}) \qquad \text{for} \quad k \in \{1, \ldots, N\}, k|L \text{ and } i \in \{1, \cdots, k\}.$$

Our goal is to determine k. The condition for a variable $\mathcal{P}_k^{(i)}$ to belong to $\mathcal{A}_{\mathrm{LV}}^0$,

$$\left\{V_n, \mathcal{P}_k^{(i)}\right\} = 0 \qquad \text{for} \quad n \in \mathbb{Z}/L\mathbb{Z}$$

reduces to

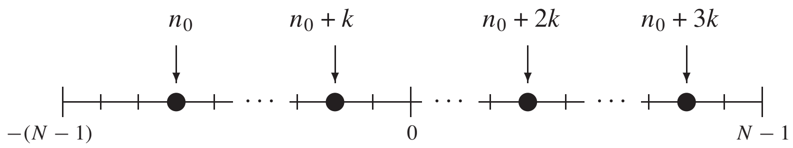
$$\sum_{m \in \mathbb{Z}/L\mathbb{Z}, m=i \bmod k} \sum_{l=1}^{N-1} (\delta_{m,n+l} - \delta_{m,n-l}) = 0. \tag{B.1}$$

Assume that we have $2j$ non-zero terms in the summation of (B.1) for $j \in \{1, \ldots, N-1\}$, where j of them offer +1 and the others offer -1. In the case of $j = 1$ we easily obtain $k = N$ if $N|L$ is satisfied, and $k = N-1$ if $(N-1)|L$. In the case of $j = N-1$ we have $k = 1$ for all L. In the following, we study the cases of $2 \leqslant k \leqslant N-2$.

Without limiting the generality, we consider the $n = 0$ case in (B.1). Let $m = n_0$ in (B.1) be the left-most lattice point where the first -1 occurs for $-(N-1) \leqslant n_0 \leqslant -N+k$. In $j = 2$ case, the condition for k (B.1) is reduced to

$$n_0 + k < 0 \qquad \text{and} \qquad N - k \leqslant n_0 + 3k \leqslant N - 1. \tag{B.2}$$

This situation is depicted as



.

Here black circles mean where the non-zero terms are offered in (B.1). We have two critical cases for n_0:

(i) when $n_0 = -(N-1)$, (B.2) reduces to

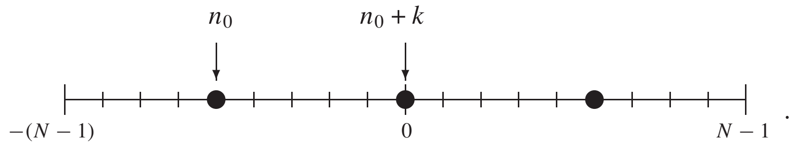
$$\frac{2N-1}{4} \leqslant k \leqslant \frac{2(N-1)}{3}. \tag{B.3}$$

(ii) When $n_0 = -N + k$, (B.2) becomes

$$\frac{2N}{5} \leqslant k \leqslant \frac{2N-1}{4}. \tag{B.4}$$

Since $\frac{2N-1}{4} \notin \mathbb{Z}$, (B.3) and (B.4) are not satisfied at the same time. When k satisfies (i), we should relate this k to a condition

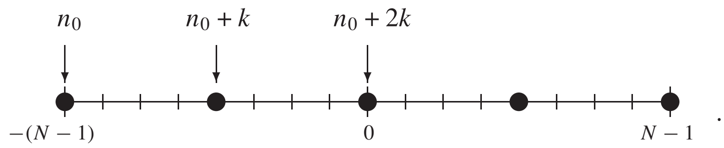
(i′) when $n_0 = -N + k$, $n_0 + k = 0$ is imposed



Then we obtain $k = \frac{N}{2}$, which turns out to be the $j = 1$ case.

On the other hand, when k satisfies (ii), we relate it to

(ii′) when $n_0 = -(N-1)$, $n_0 + 2k = 0$ is required



Therefore we obtain $k = \frac{N-1}{2}$, which is a special case of $j = 2$.

The conditions (i) and (i′) do not contradict each other for $N \geqslant 4$, and so do not (ii) and (ii′) for $N \geqslant 5$. Then we obtain $k = \frac{N}{2}$ (resp. $\frac{N-1}{2}$) if $2|N$ (resp. $2|(N-1)$).

In general $j_{\geqslant 3}$ cases, (B.1) reduces to

$$n_0 + (j-1)k < 0 \qquad \frac{N - n_0}{2j} \leqslant k \leqslant \frac{N - 1 - n_0}{2j - 1}. \tag{B.5}$$

Then two critical cases are written as follows:

(i) when $n_0 = -(N-1)$, (B.5) becomes

$$\frac{2N-1}{2j} \leqslant k \leqslant \frac{2(N-1)}{2j-1}.$$

And when $n_0 = -N + k$, $n_0 + (j-1)k = 0$. Then we obtain $k = \frac{N}{j}$ for $N \geqslant 2j$ and $j|N$.

(ii) When $n_0 = -N + k$,

$$\frac{2N}{2j+1} \leqslant k \leqslant \frac{2N-1}{2j}.$$

And when $n_0 = -(N-1)$, $n_0 + jk = 0$. Then we get $k = \frac{N-1}{j}$ for $N \geqslant 2j + 1$ and $j|(N-1)$.

Finally we obtain the set $\mathcal{K}$ (4.7) k belongs to. $\square$

Appendix C

Proof of proposition 4.5. We show the first part of proposition 4.5 in more general setting. Assume that $\mathbf{T}_{\mathrm{LV}}(z)$ has a form as

$$\mathbf{T}_{\mathrm{LV}}(z) = \boldsymbol{\mu}_0^{\mathrm{LV}} z^m + \boldsymbol{\mu}_1^{\mathrm{LV}} z^{m-1} + \cdots + \boldsymbol{\mu}_m^{\mathrm{LV}}.$$

Let all matrix elements of $\boldsymbol{\mu}_i^{\mathrm{LV}}$ belong to $\mathcal{A}'_{\mathrm{LV}}$. We relate $\mathbf{T}_{\mathrm{LV}}(z)$ to a matrix $\mathbf{T}(z)$ by the gauge transformation

$$\mathbf{T}(z) = \mathbf{B}\mathbf{T}_{\mathrm{LV}}(z)\mathbf{B}^{-1}.$$

Here the gauge matrix $\mathbf{B}$ is a diagonal matrix independent of z, whose entries belong to a Poisson bracket algebra where $\mathcal{A}'_{\mathrm{LV}}$ is embedded. Then the matrix $\mathbf{T}(z)$ has a similar form to $\mathbf{T}_{\mathrm{LV}}(z)$,

$$\mathbf{T}(z) = \boldsymbol{\mu}_0 z^m + \boldsymbol{\mu}_1 z^{m-1} + \cdots + \boldsymbol{\mu}_m$$

where $\boldsymbol{\mu}_i = \mathbf{B}\boldsymbol{\mu}_i^{\mathrm{LV}}\mathbf{B}^{-1}$. With these settings we have

Proposition 4.5′. *Let $\mathcal{A}_N$ be a Poisson bracket algebra generated by the entries of a matrix $\mathbf{N}(z)$ related to $\mathbf{T}(z)$ by an invertible matrix $\mathbf{S}$ as*

$$\mathbf{N}(z) = \mathbf{S}\mathbf{T}(z)\mathbf{S}^{-1} \qquad \mathbf{S} = \begin{pmatrix} \vec{e}_i \boldsymbol{\mu}^{(1)} \\ \vec{e}_i \boldsymbol{\mu}^{(2)} \\ \vdots \\ \vec{e}_i \boldsymbol{\mu}^{(N)} \end{pmatrix}.$$

Here each of $\boldsymbol{\mu}^{(i)}$ is a product of $\boldsymbol{\mu}_j (j = 0, \ldots, m)$. Then $\mathcal{A}_N$ is embedded in $\mathcal{A}'_{\mathrm{LV}}$.

Proof. It is sufficient to show that the matrix elements of $\mathbf{N}(z)$ belong to $\mathbb{C}(\mathcal{P}_0, V_n; n \in \mathbb{Z}/L\mathbb{Z})$. Using $\mathbf{B} = \mathrm{diag}[\mathrm{b}_1, \mathrm{b}_2, \ldots, \mathrm{b}_{\mathrm{N}}]$, the matrix $\boldsymbol{\mu}^{(i)}$ is rewritten as

$$\boldsymbol{\mu}^{(i)} = \mathbf{B}\boldsymbol{\mu}^{(i)\mathrm{LV}}\mathbf{B}^{-1}$$

where $\boldsymbol{\mu}^{(i)\mathrm{LV}}$ is the associated product of $\boldsymbol{\mu}_j^{\mathrm{LV}}$. Therefore the gauge matrix $\mathbf{S}$ can be written as

$$\mathbf{S} = b_i \mathbf{S}_{\mathrm{LV}}\mathbf{B}^{-1} \qquad \mathbf{S}_{\mathrm{LV}} = \begin{pmatrix} \vec{e}_i \boldsymbol{\mu}^{(1)\mathrm{LV}} \\ \vec{e}_i \boldsymbol{\mu}^{(2)\mathrm{LV}} \\ \vdots \\ \vec{e}_i \boldsymbol{\mu}^{(N)\mathrm{LV}} \end{pmatrix}.$$

Then $\mathbf{N}(z)$ is obtained as

$$\begin{aligned} \mathbf{N}(z) &= b_i \mathbf{S}_{\mathrm{LV}}\mathbf{B}^{-1}\mathbf{T}(z)\mathbf{B}\mathbf{S}_{\mathrm{LV}}^{-1} b_i^{-1} \\ &= \mathbf{S}_{\mathrm{LV}}\mathbf{T}_{\mathrm{LV}}(z)\mathbf{S}_{\mathrm{LV}}^{-1}. \end{aligned}$$

Since all entries of $\mathbf{S}_{\mathrm{LV}}$ and $\mathbf{T}_{\mathrm{LV}}(z)$ belong to $\mathcal{A}'_{\mathrm{LV}}$, the proposition follows. □

When we apply this proposition to the case $\mathbf{T}(z) = \mathbf{T}_{m;n_1,n_2}(z)$, the first part (i) follows. Further, from (i) we see that the separation equation (2.22) can be written in terms of entries in $\mathbf{M}_F(z)$, then we obtain the second part (ii). □

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